Introducing the Law of Large Numbers to Statistics Courses Through an Interactive Programming Activity

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August 4, 2023

Two Kinds of Probability

- Empirical Probability: The probability of an event A is the number of times A occurs divided by the number of repetitions, n.

- Theoretical Probability: The probability of the occurrence of an event that comes from a sample space of known equally favourable outcomes.

Two Kinds of Probability

Scenario

Suppose I flip a fair and balanced coin 10 times.

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$$H, T, H, H, T, H, H, T, H, H$$

What's the probability of heads?

- Empirical Probability:

Two Kinds of Probability

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Suppose I flip a fair and balanced coin 10 times. I get the following sequence of flips:

$$H, T, H, H, T, H, H, T, H, H$$

What's the probability of heads?

- Empirical Probability: We have seven heads out of ten flips. So, the probability of heads can be calculated as $P(H) = \frac{7}{10} = .7$
- Theoretical Probability:

Two Kinds of Probability

Scenario

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$$H, T, H, H, T, H, H, T, H, H$$

What's the probability of heads?

- Empirical Probability: We have seven heads out of ten flips. So, the probability of heads can be calculated as $P(H) = \frac{7}{10} = .7$
- Theoretical Probability: We have a fair and balanced coin. We can only have a Heads or a Tails. Thus, the probability of heads should be $P(H) = \frac{1}{2} = .5$

We don't get the same answer!

The Law of Large Numbers

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The Law of Large Numbers (LLN)

As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome.

In other words, the empirical probability gets closer and closer to the theoretical probability as n increases!

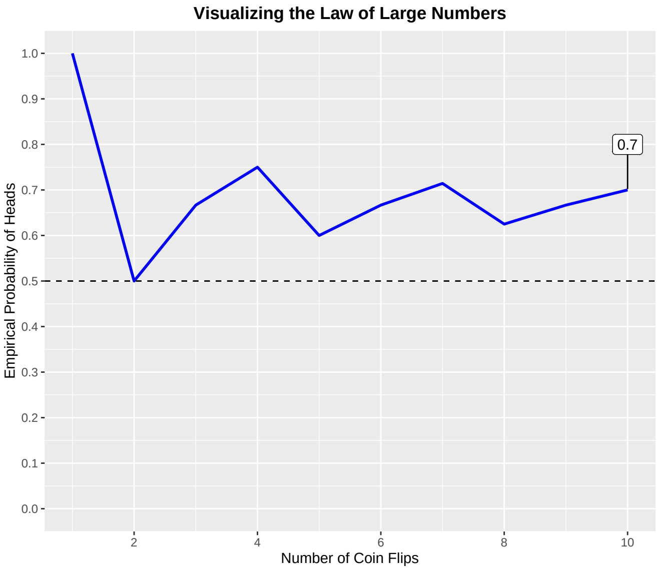
Setup

- We'll use the R programming language to visualize the LLN in action.
- Let's simulate a fair and balanced coin being flipped a number of times.
- What will happen to the empirical probability of a heads as n increases?

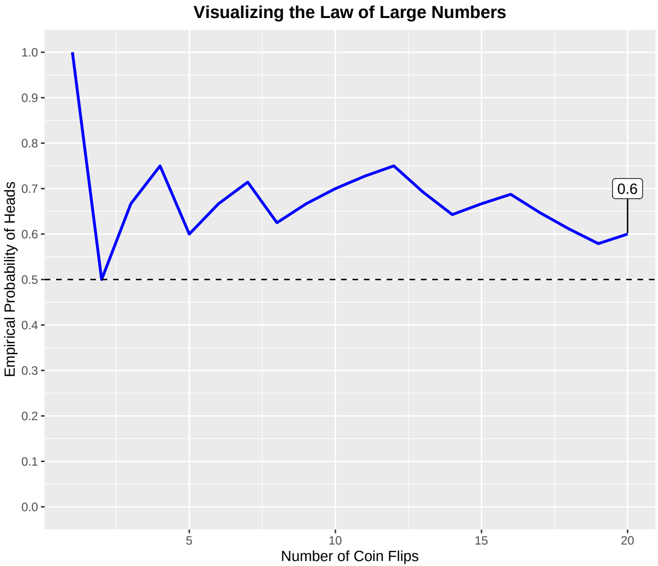
Setup

- We'll use the R programming language to visualize the LLN in action.
- Let's simulate a fair and balanced coin being flipped a number of times.
- What will happen to the empirical probability of a heads as n increases?
- Let's start with 10 flips. Earlier, we got the following sequence $H, T, H, H, T, H, H, T, H, H$.
- This leads to an empirical probability of .7.

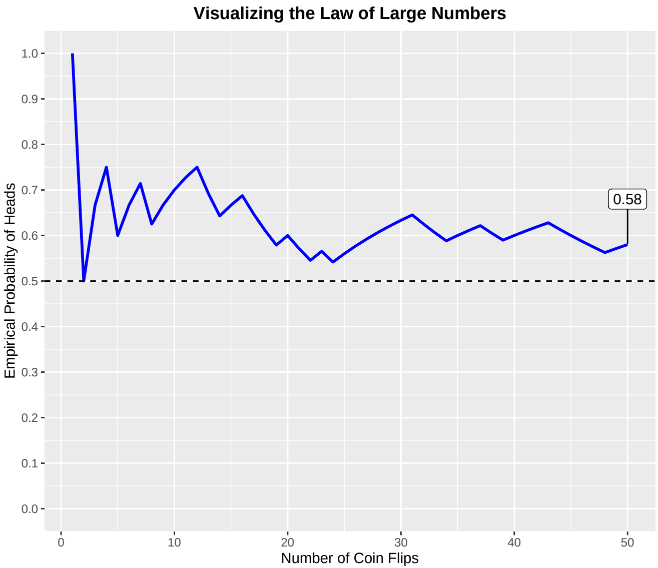
10 Flips


Visualizing the Law of Large Numbers
Empirical Probability of Heads
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
2
4
6
8
10
Number of Coin Flips
0.7

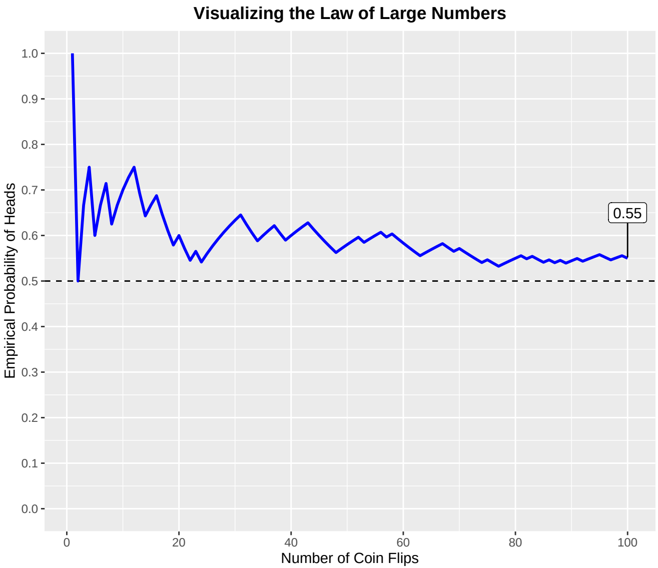
20 Flips


Visualizing the Law of Large Numbers
Empirical Probability of Heads
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.6
5
10
15
20
Number of Coin Flips

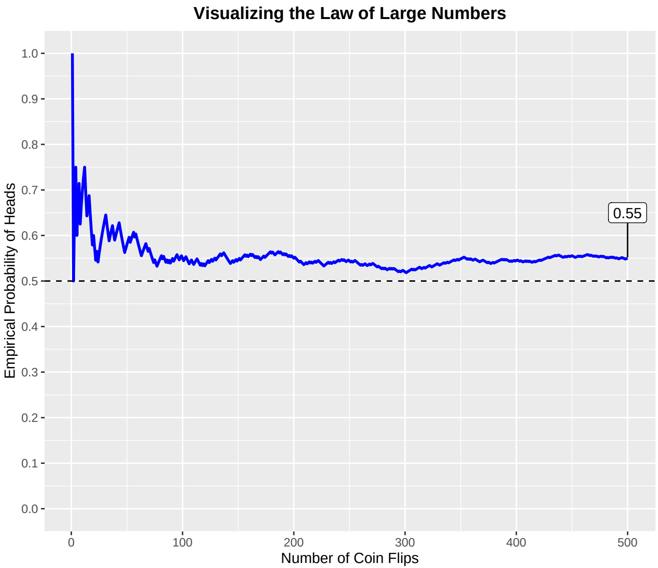
50 Flips


Visualizing the Law of Large Numbers
Empirical Probability of Heads
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
10
20
30
40
50
Number of Coin Flips
0.58

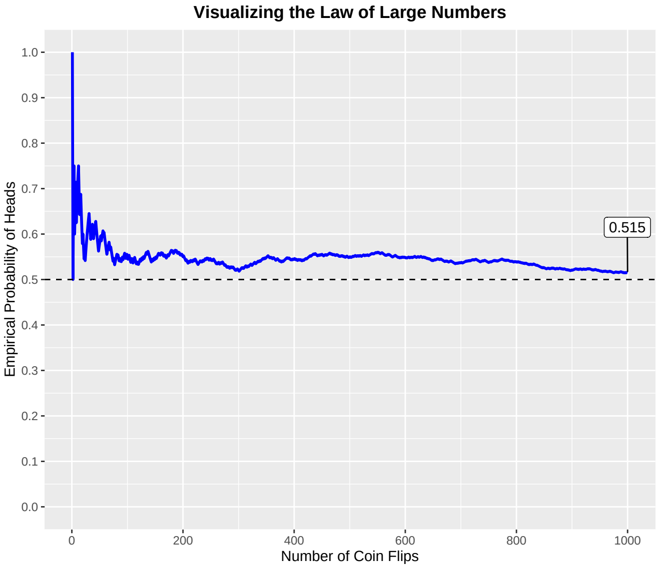
100 Flips


Visualizing the Law of Large Numbers
Empirical Probability of Heads
Number of Coin Flips
0.55
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
20
40
60
80
100

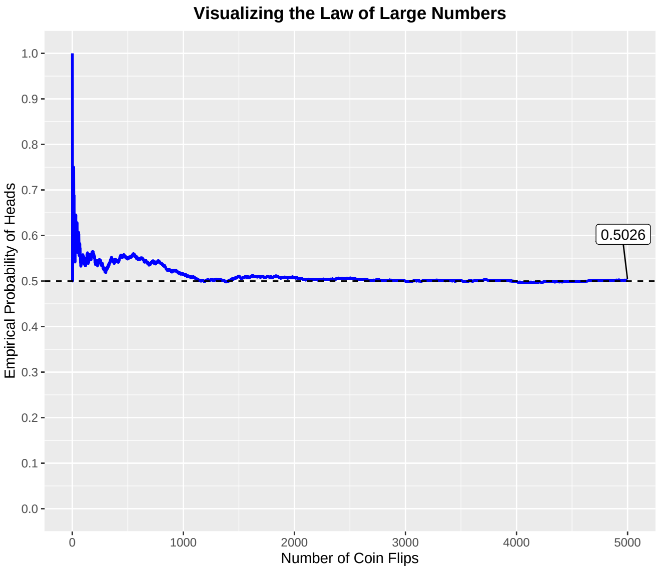
500 Flips


Visualizing the Law of Large Numbers
Empirical Probability of Heads
Number of Coin Flips
0.55
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
100
200
300
400
500

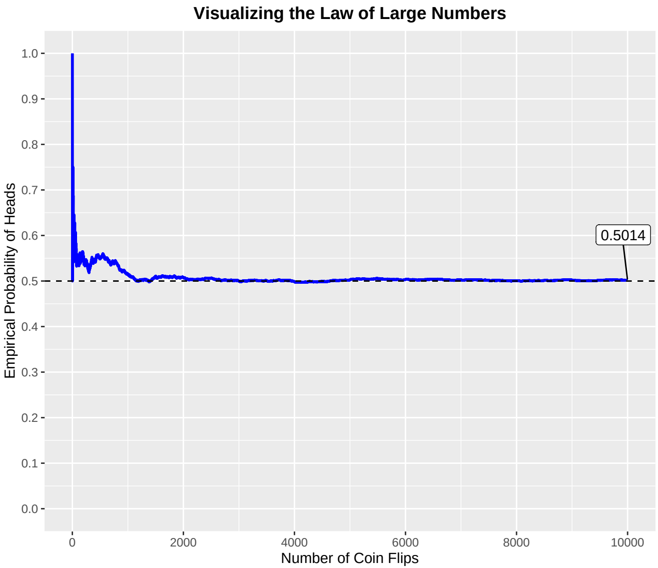
1000 Flips


Visualizing the Law of Large Numbers
Empirical Probability of Heads
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.515
0
200
400
600
800
1000
Number of Coin Flips

5000 Flips


Visualizing the Law of Large Numbers
Empirical Probability of Heads
Number of Coin Flips
0.5026
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1000
2000
3000
4000
5000

10,000 Flips


Visualizing the Law of Large Numbers
Empirical Probability of Heads
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
2000
4000
6000
8000
10000
Number of Coin Flips
0.5014

Multiple People

Setup

- The Law of Large Numbers can be used for all sorts of fun applications. Let's have some fun simulating die rolls. Let's pretend we're playing Dungeons and Dragons. We'll roll a 20-sided die (D20) many times.

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- Rolling a 1 is considered a critical failure. If a 1 shows up a lot more often than the other numbers, you're considered to have a cursed die. Personally, a 1 always seems to show up at the worst possible moments.

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- The Law of Large Numbers can be used for all sorts of fun applications. Let's have some fun simulating die rolls. Let's pretend we're playing Dungeons and Dragons. We'll roll a 20-sided die (D20) many times.
- Rolling a 1 is considered a critical failure. If a 1 shows up a lot more often than the other numbers, you're considered to have a cursed die. Personally, a 1 always seems to show up at the worst possible moments.
- Let's determine if we have a fair die or a cursed die by charting the empirical probability of rolling a 1. If we have a cursed die, let's try to determine the theoretical probability of rolling a 1 with our die. If we have a fair D20, then the probability of rolling a 1 will be $\frac{1}{20} = .05$.

Setup

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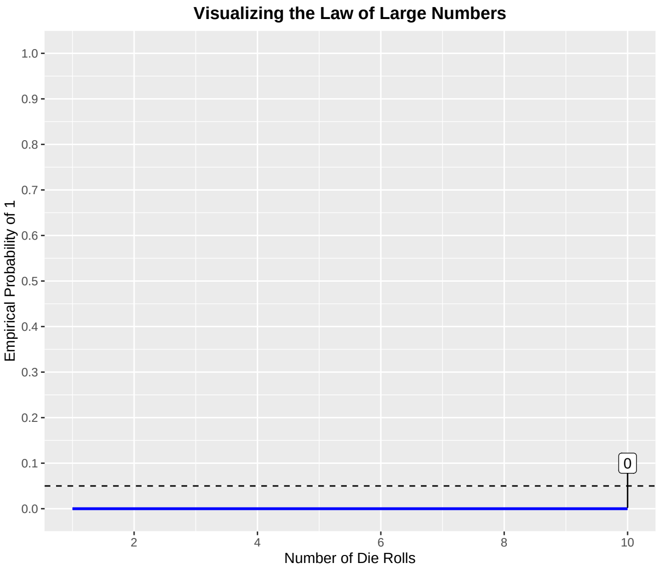
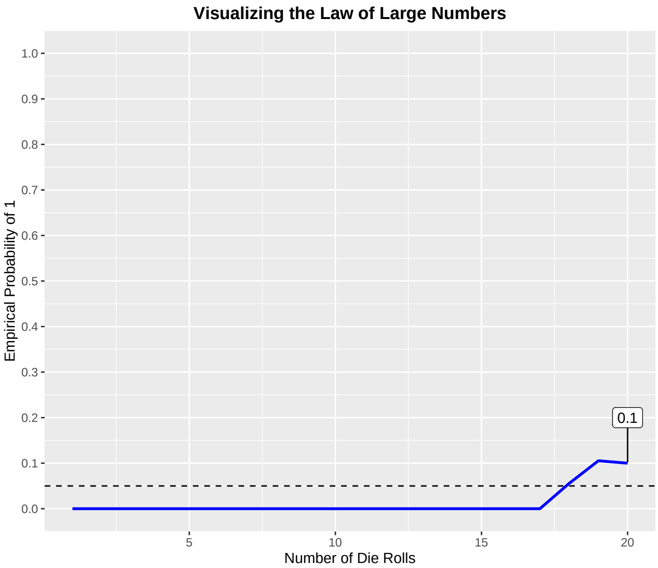
- Let's determine if we have a fair die or a cursed die by charting the empirical probability of rolling a 1. If we have a cursed die, let's try to determine the theoretical probability of rolling a 1 with our die. If we have a fair D20, then the probability of rolling a 1 will be $\frac{1}{20} = .05$.
- How many rolls will it take us to pretty accurately determine this?

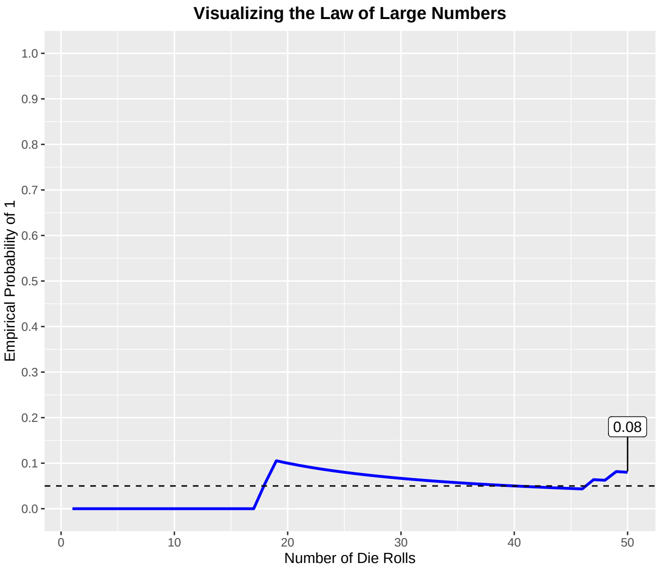
10 Rolls


Visualizing the Law of Large Numbers
Empirical Probability of 1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
2
4
6
8
10
Number of Die Rolls

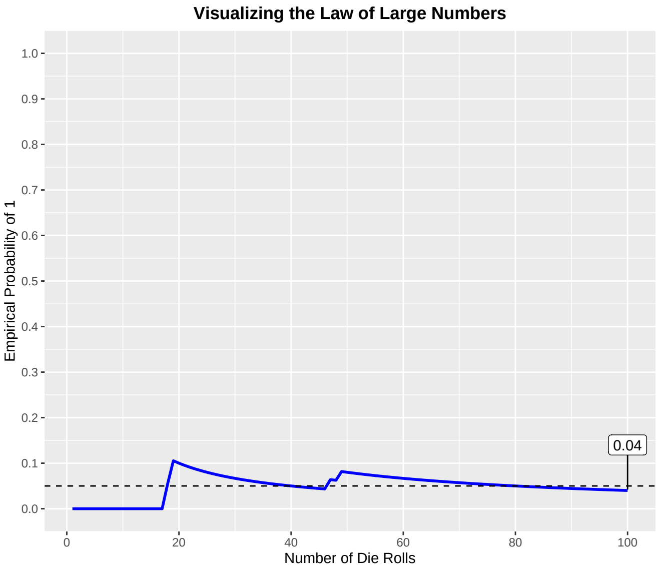
20 Rolls


Visualizing the Law of Large Numbers
Empirical Probability of 1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.1
5
10
15
20
Number of Die Rolls

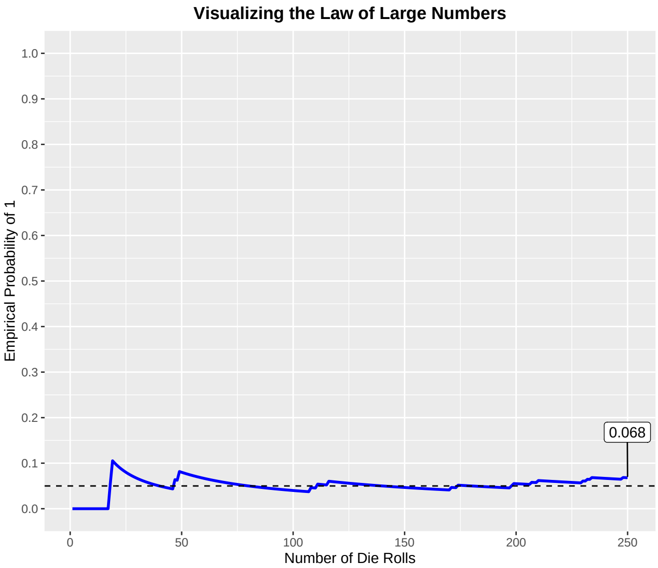
50 Rolls


Visualizing the Law of Large Numbers
Empirical Probability of 1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
10
20
30
40
50
Number of Die Rolls
0.08

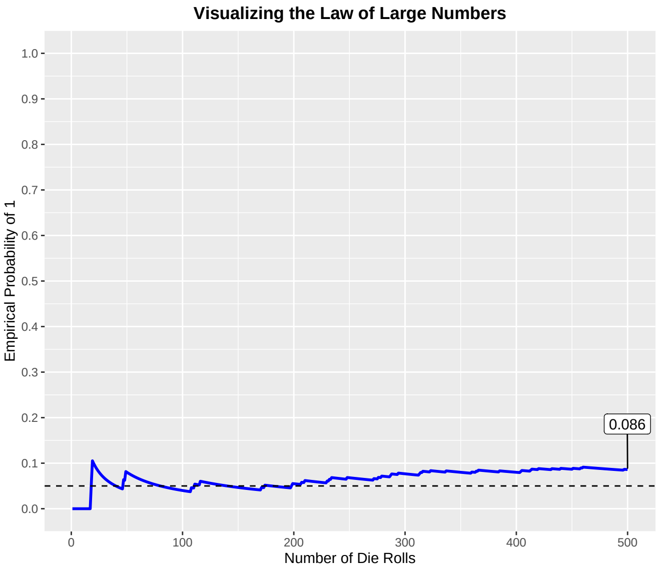
100 Rolls


Visualizing the Law of Large Numbers
Empirical Probability of 1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
20
40
60
80
100
Number of Die Rolls
0.04

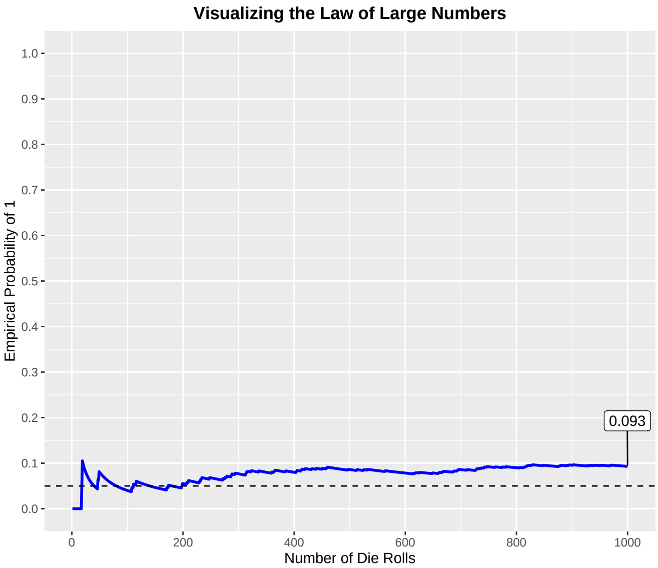
250 Rolls


Visualizing the Law of Large Numbers
Empirical Probability of 1
Number of Die Rolls
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
50
100
150
200
250
0.068

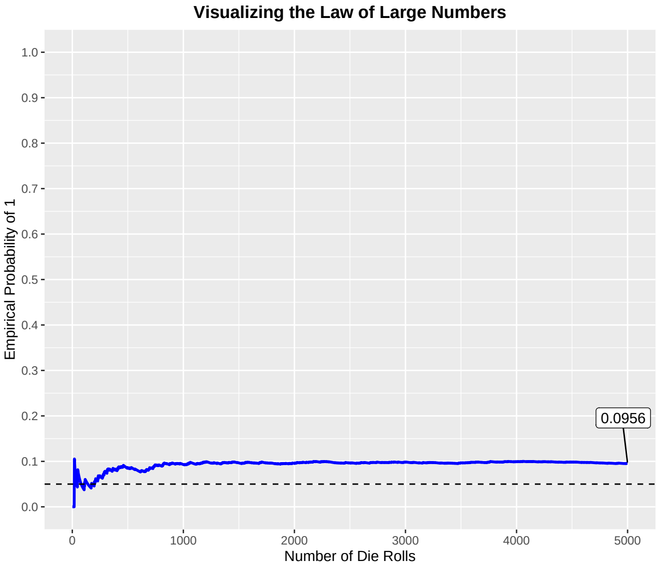
500 Rolls


Visualizing the Law of Large Numbers
Empirical Probability of 1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
100
200
300
400
500
Number of Die Rolls
0.086

1000 Rolls


Visualizing the Law of Large Numbers
Empirical Probability of 1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.093
0
200
400
600
800
1000
Number of Die Rolls

5000 Rolls


Visualizing the Law of Large Numbers
Empirical Probability of 1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0956
0
1000
2000
3000
4000
5000
Number of Die Rolls

Activity in the Classroom

I show this activity to all of my statistics classes during the probability chapter.

We run through the activity together live as a class. The class determines the seed, how many flips to run through, etc. We then talk through what's happening as n increases.

Thank You!